Detecting genuine multipartite correlations in terms of the rank of coefficient matrix

Bo Li,[1,2,*] Leong Chuan Kwek,[3] and Heng Fan[2,†]

[1] *Department of Mathematics and Computer, Shangrao Normal University, Shangrao 334001, China*
[2] *Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China*
[3] *Center for Quantum Technologies, National University of Singapore, 2 Science Drive 3, Singapore 117543 and National Institute of Education and Institute of Advanced Studies and Institute of Advanced Studies, Nanyang Technological University, Singapore, 637616.*

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We propose a method to detect genuine quantum correlation for arbitrary quantum state in terms of the rank of coefficient matrices associated with the pure state. We then derive a necessary and sufficient condition for a quantum state to possess genuine correlation, namely that all corresponding coefficient matrices have rank larger than one. We demonstrate an approach to decompose the genuine quantum correlated state with high rank coefficient matrix into the form of product states with no genuine quantum correlation for pure state.

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INTRODUCTION

Quantification of multipartite entanglement has remained an outstanding but important challenge in quantum information science. Besides applications in quantum information science, multipartite states have served as crucial resources for a myriad of quantum information processing tasks ranging from quantum cryptography and secret sharing [1, 2] to quantum simulation [3], measurement-base quantum computation [4] and high precision metrology [5].

While bipartite entanglement is well understood, genuine multipartite entangled states are less studied. Genuine n-partite entanglement needs entanglement of all n constituent systems. It should not be separable for any partition of the system. Moreover, it turns out that genuine multipartite entanglement are needed in several quantum information processes. Thus it is important to detect not just multipartite entanglement [6–11] but also genuine multipartite entanglement. However, detection of genuine multipartite entangled state experimentally however has proved elusive.

Bennett[12] *et al.* have recently proposed some reasonable postulates for measures of genuine multipartite correlations. They gave the following postulates or definitions:

Definition 1. A state of n particles has genuine n-partite correlations if it is not a product state in every bipartite cut.

Definition 2. A state of n particles has genuine k-partite correlations if there exists a k-particle subset whose reduced state has genuine k-partite correlations.

Definition 3. A state has degree of correlations equal to n if there exists a subset of n particles which has genuine n-partite correlations and there does not exist a subsets of m particles which has genuine m-partite correlations for any $m > n$.

The above definition is slightly different from the genuine multipartite entanglement, i.e., for pure state, a state has genuine n-partite correlation if and only if it has genuine entanglement. For mixed state, however, a state has genuine n-partite entanglement if it is not a mixture of pure states that do not have genuine n-partite entanglement. Comparing the definitions, it is not difficult to find that the set of genuine correlated states that contains the set of genuine entanglement states for mixed states.

Very recently, Giorgi[13] *et al.* introduced a measure of genuine total, classical, and quantum correlations based on the use of the relative entropy to quantify the distance between the studied and the corresponding benchmark states; Rulli[14–16] *et al.* introduce some measure of global quantum discord and geometry discord. Beside these, very little is known to us on multipartite quantum correlation. In this paper, we provide a necessary and sufficient condition for detecting genuine multipartite correlations for arbitrary n-qubit quantum system in terms of the coefficient matrices of the pure state. Then we demonstrate an approach to decompose a state into the product of genuine correlated states.

COEFFICIENT MATRIX AND GENUINE MULTIPARTITE CORRELATION FOR PURE n-QUBIT STATE.

We first review the construction of coefficient matrix as introduced in Ref.[17, 18]. Let $|\psi\rangle_{1\cdots n} = \sum_{i=0}^{2^n-1} a_i|i\rangle$ be an n-qubit pure state. We rewrite $|\psi\rangle_{1\cdots n}$ as $|\psi\rangle_{1\cdots n} = \sum_{j=0}^{2^{n-\ell}-1}\sum_{i=0}^{2^\ell-1} a_{ij}|i\rangle\otimes|j\rangle$. A coefficient matrix associated to the state $|\psi\rangle_{1\cdots n}$ is given by

$$C_{1\cdots\ell,(\ell+1)\cdots n}(|\psi\rangle_{1\cdots n}) = (a_{ij})_{2^\ell\times 2^{n-\ell}}. \tag{1}$$

We remark that the splitting of $\ell : (n-\ell)$ is arbitrary so ℓ can take any possible values. For more general case, we designate a permutation $\{q_1, q_2, \cdots, q_n\}$ of $\{1, 2, \cdots, n\}$, and $C_{q_1\cdots q_\ell,q_{\ell+1}\cdots q_n}(|\psi\rangle_{1\cdots n})$ be the coefficient matrix corresponding to the permutation constructed from $C_{12\cdots\ell,\ell+1\cdots n}(|\psi\rangle_{1\cdots n})$ in Eq. (1). For convenience, we omit the subscripts $q_{\ell+1},\cdots,q_n$ and simply write $C_{q_1\cdots q_\ell}$, whenever the column bits are clear from the context. If $|\psi\rangle_{1\cdots n}$ is a bipartite product state, i.e., $|\psi\rangle_{1\cdots n} = |\phi\rangle_{q_1\cdots q_\ell}\otimes|\phi'\rangle_{q_{\ell+1}\cdots q_n}$, by the construction of the coefficient matrix, we have the rank of $C_{q_1\cdots q_\ell}(|\psi\rangle_{1\cdots n})$ must be $r(C_{q_1\cdots q_\ell}(|\psi\rangle_{1\cdots n})) = 1$. Together with definition 1 , we obtain a very easy criterion to detect the genuine multipartite correlation for pure n-qubit state.

Theorem 1. *A pure n-qubit state $|\psi\rangle_{1\cdots n}$ has genuine n-partite correlations if and only if for every permutation $\{q_1, q_2, \cdots, q_n\}$ and $0 < \ell < n$, the rank of the coefficient matrix has $r(C_{q_1\cdots q_\ell}(|\psi\rangle_{1\cdots n})) \neq 1$.*

To show the power of Theorem 1, we first give an example to show that the n-qubit symmetric Dicke states $|\ell, n\rangle$ with $\ell(0 < \ell < n)$ excitations [20]

$$|\ell, n\rangle = \binom{n}{\ell}^{-1/2} \sum_k P_k |1_1, 1_2, \cdots, 1_\ell, 0_{\ell+1}, \cdots, 0_n\rangle, \quad (2)$$

have genuine correlations, where $\{P_k\}$ is the set of all distinct permutation of the spins. It is shown in [17] that the rank of $|\ell, n\rangle$ is $\ell + 1$, for $0 < \ell < n$, we have $r(C_{q_1\cdots q_\ell}(|\ell, n\rangle)) \geq 2$ for any permutation. By Theorem 1, we have that any n-qubit symmetric Dicke states $|\ell, n\rangle$ with $\ell(0 < \ell < n)$ have genuine correlations, for $\ell = 0$ or $\ell = n$, we have $|0, n\rangle$ and $|n, n\rangle$ are both product state.

We now further consider arbitrary Permutation symmetric pure n-qubit state, by using Vieta's formulas, any symmetric pure n-qubit state can be written into the combination of symmetric Dicke states[19],

$$|\psi\rangle_{1\cdots n} = \sum_{\ell=0}^{n} a_\ell |\ell, n\rangle, \quad (3)$$

we first state that if there exist $\ell(0 < \ell < n)$ such that $a_\ell^2 \neq a_{\ell+1}a_{\ell-1}$, then $|\psi\rangle_{1\cdots n}$ in (3) must have genuine correlation. In fact, for any permutation, one selects $i_2 \cdots i_{n-1}$ so that $\ell - 1$ bits are equal to 1 and the rest of bits are equal to 0. The coefficients of $|0i_2 \cdots i_{n-1}1\rangle$ and $|1i_2 \cdots i_{n-1}0\rangle$ are a_ℓ, the coefficient of $|0i_2 \cdots i_{n-1}0\rangle$ is $a_{\ell-1}$, and that of $|1i_2 \cdots i_{n-1}1\rangle$ is $a_{\ell+1}$. We can find that the foregoing coefficients constitute a 2×2 nonzero minor of the coefficient matrix. Therefore, we have $r(C_{q_1\cdots q_\ell}(|\psi\rangle_{1\cdots n})) \geq 2$, and the state has genuine multipartite correlation. If we have $a_\ell^2 = a_{\ell+1}a_{\ell-1}$ for all $\ell(0 < \ell < n)$, that is $\frac{a_0}{a_1} = \frac{a_1}{a_2} = \cdots = \frac{a_{n-1}}{a_n} \equiv \alpha$, then $|\psi\rangle_{1\cdots n}$ is given as

$$\begin{aligned} |\psi\rangle_{1\cdots n} &= a_n \sum_{\ell=0}^{n} \alpha^{n-\ell} |\ell, n\rangle \\ &= a_n(\alpha|0\rangle + |1\rangle) \otimes \cdots \otimes (\alpha|0\rangle + |1\rangle), \end{aligned} \quad (4)$$

which is a product state. In summary, we find that an arbitrary symmetric pure n-qubit states have genuine correlations except the state is in one of the following three cases: $|0, n\rangle$, $|n, n\rangle$, $|\psi\rangle_{1\cdots n}$ in Eq. (4). Thus, we have completely describe the genuine correlations for arbitrary symmetric pure n-qubit states.

Using Theorem 1, we can decompose a pure state into a product of genuine correlated states. For any pure state $|\psi\rangle_{1\cdots n}$, if there exists a permutation $\{q_1, q_2, \cdots, q_n\}$ and $0 < \ell < n$ such that $r(C_{q_1\cdots q_\ell}) = 1$, then we have the decomposition $|\psi\rangle_{1\cdots n} = |\phi\rangle_{q_1\cdots q_\ell} \otimes |\phi'\rangle_{q_{\ell+1}\cdots q_n}$. We can further decompose $|\phi\rangle_{q_1\cdots q_\ell}, |\phi'\rangle_{q_{\ell+1}\cdots q_n}$ into a product of some other states, and eventually obtain that $|\psi\rangle_{1\cdots n}$ be the product of some genuine correlated states. It is not difficult to discern that for any pure state, regardless of the permutation, the decomposition to genuine correlated states is unique[22]. By definitions 2 and 3, for any pure state, we immediately have the following theorem.

Theorem 2. *Suppose that $|\psi\rangle_{1\cdots n}$ has the decomposition $|\psi\rangle_{1\cdots n} = |\psi\rangle^{(m_1)} \otimes \cdots \otimes \psi\rangle^{(m_M)}$, and $|\psi\rangle_{1\cdots n}$ has genuine $m_1, \cdots, m_M$-partite correlations, then the degree of correlations is $\max\{m_1, \cdots, m_M\}$, here $|\psi\rangle^{(m_1)}, \cdots, \psi\rangle^{(m_M)}$ are states of $m_1, \cdots, m_M$-particles genuine correlated states, and $n = m_1 + \cdots + m_M$.*

GENUINE MULTIPARTITE CORRELATION FOR ARBITRARY n-QUBIT MIXED STATE

We next consider the detection of genuine correlations for mixed state. Suppose a n-particle state ρ has a bipartite cut $\rho = \rho_1 \otimes \rho_2$, with $\rho_1 = \sum_{i=1}^{r} \lambda_i |\psi_i\rangle\langle\psi_i|$, $\rho_2 = \sum_{j=1}^{s} \mu_j |\phi_j\rangle\langle\phi_j|$ be the spectral decomposition. The density matrices ρ_1, ρ_2 are the n_1, n_2-particle density matrix respectively($n_1 + n_2 = n$), and $\lambda_i\mu_j, |\psi_i\rangle \otimes |\phi_j\rangle$ are the eigenvalues and eigenvectors of ρ, the rank of ρ is $r \times s$. We can regard ρ_1 as a reduced state of a pure $(n_1 + r)$-qubit state

$$|\psi\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i} |\psi_i\rangle \otimes |\underbrace{0 \cdots 1_i \cdots 0}_{r}\rangle. \quad (5)$$

Similarly, ρ_2 can be seen as a reduced state of a pure $(n_2 + s)$-qubit state

$$|\phi\rangle = \sum_{j=1}^{s} \sqrt{\mu_j} |\phi_j\rangle \otimes |\underbrace{0 \cdots 1_j \cdots 0}_{s}\rangle. \quad (6)$$

Thus, ρ can be seen as a reduced state of a pure (n_1+n_2+r+s)-qubit state

$$\begin{aligned} |\Phi\rangle &= \sum_{i=1}^{r} \sqrt{\lambda_i} |\psi_i\rangle |\underbrace{0 \cdots 1_i \cdots 0}_{r}\rangle \otimes \\ &\quad \sum_{j=1}^{s} \sqrt{\mu_j} |\phi_j\rangle |\underbrace{0 \cdots 1_j \cdots 0}_{s}\rangle \\ &= |\psi\rangle \otimes |\phi\rangle. \end{aligned} \quad (7)$$

For a generic mixed state ρ, we provide the following process to detect the genuine correlations. We first give the spectral decomposition of ρ,

$$\rho = \sum_{i=1}^{R} \lambda_i |\Phi_i\rangle\langle\Phi_i|, \quad (8)$$

for every factors a, b of R supply $R = a \times b$, we construct a pure $(n + a + b)$-qubit state

$$\begin{aligned} |\Phi\rangle^{ab} &= \sum_{i=1}^{a} \sum_{j=1}^{b} \sqrt{\lambda_{(i-1)\times b+j}} |\Phi_{(i-1)\times b+j}\rangle \otimes |\underbrace{0 \cdots 1_i \cdots 0}_{a}\rangle \\ &\quad \otimes |\underbrace{0 \cdots 1_j \cdots 0}_{b}\rangle. \end{aligned} \quad (9)$$

One sees that ρ is a reduced state of $|\Phi\rangle^{ab}$ by tracing out the last $(a+b)$-partite. If ρ has a bipartite cut with rank $a\times b$, Eq. (9) must take the form of Eq. (7), and since $|\Phi\rangle$ in Eq. (7) is a product state, we have a permutation $\{q_1, q_2, \cdots, q_n\}$ of $\{1, 2, \cdots, n\}$, such that the rank of coefficient matrix of $|\Phi\rangle^{ab}$ has $r(C_{q_1\cdots q_\ell n+1\cdots n+a}(|\Phi\rangle^{ab})) = 1$, here $0 < \ell < n$. On the contrary, if there exists a permutation $\{q_1, q_2, \cdots, q_n\}$ and a, b such that the rank of the coefficient matrix of $|\Phi\rangle^{ab}$ in Eq. (9) gives $r(C_{q_1\cdots q_\ell n+1\cdots n+a}(|\Phi\rangle^{ab})) = 1$, then $|\Phi\rangle^{ab}$ can be rewritten as a $(q_1\cdots q_\ell n+1\cdots n+a)\times(q_{\ell+1}\cdots q_n n+a+1\cdots n+a+b)$-bipartite product state, by tracing out the $(a+b)$-partite, we obtain that ρ is a product of $(q_1\cdots q_\ell)$-partite by $(q_{\ell+1}\cdots q_n)$-partite. Thus, together with definition 1 given by Bennett *et al.*, we have proved the following theorem.

Theorem 3. *Suppose a n-particle state ρ has rank R, the spectral decomposition of ρ is given by $\rho = \sum_{i=1}^{R}\lambda_i|\Phi_i\rangle\langle\Phi_i|$, for every factors a, b of R supply $R = a\times b$, construct $(n+a+b)$-particle pure state $|\Phi\rangle^{ab}$ in the form of Eq. (9), then ρ has genuine n-particle correlation if and only if for every a, b, permutation $\{q_1, q_2, \cdots, q_n\}$, and $0<\ell<n$, the rank of the coefficient matrix of $|\Phi\rangle^{ab}$ gives rise to $r(C_{q_1\cdots q_\ell n+1\cdots n+a}(|\Phi\rangle^{ab})) \neq 1$.*

Theorem 3 provides us with a natural way to detect genuine k-partite correlations for a state ρ of n particles, and also its degree of correlations. According to the definition 2, 3, we need only check if there exists a subset $\{q_1\cdots q_k\}$ of $\{1, 2, \cdots, n\}$ such that each reduced density matrix with $\{q_1\cdots q_k\}$ particles, $\rho_{q_1\cdots q_k}$, has genuine multipartite correlations to confirm that ρ has genuine k-particle correlations, and the maximum k value is just the degree of correlations.

We show a flow chart in Fig. 1 to demonstrate the decomposition of an arbitrary n-qubit quantum state into a product of genuine k correlated states. From the flowchart, we see that we can use an algorithmic approach to decompose a quantum state into a product of genuine k correlated states, and obtain its degree of correlations.

We now provide some illustrative examples concerning the usefulness of Theorem 3. Consider the n-particle state $\rho = p|\ell, n\rangle\langle\ell, n| + (1-p)|\ell', n\rangle\langle\ell', n|$, the rank of ρ is $r(\rho) = 2$, and $|\Phi\rangle^{ab}$ in Eq. (9) given by

$$|\Phi\rangle^{ab} = \sqrt{p}|\ell, n\rangle|1\rangle|10\rangle + \sqrt{1-p}|\ell', n\rangle|1\rangle|01\rangle. \quad (10)$$

Since $\ell \neq \ell'$, for any $\{q_1, \cdots, q_k\}$, we can select the term $|\alpha_1\cdots\alpha_n\rangle$ in $|\ell, n\rangle$ and $|\beta_1\cdots\beta_n\rangle$ in $|\ell', n\rangle$ which are different on $\{q_1, \cdots, q_k\}$-particles, notice that the coefficients of $|\alpha_1\cdots\alpha_k\beta_{k+1}\cdots\beta_n\rangle|1\rangle|10\rangle$, $|\beta_1\cdots\beta_k\alpha_{k+1}\cdots\alpha_n\rangle|1\rangle|01\rangle$ are zero, we have $C(|\Phi\rangle^{ab}_{q_1\cdots q_k n+1})$ has a nonzero 2×2 minor. Therefore $r(C(|\Phi\rangle^{ab}_{q_1\cdots q_k n+1})) \geq 2$ for any $0<k<n$, which means ρ has genuine multipartite correlation. As a corollary, we can also show that $\rho = p|GHZ\rangle\langle GHZ| + (1-p)|W\rangle\langle W|$ has genuine multipartite correlation.

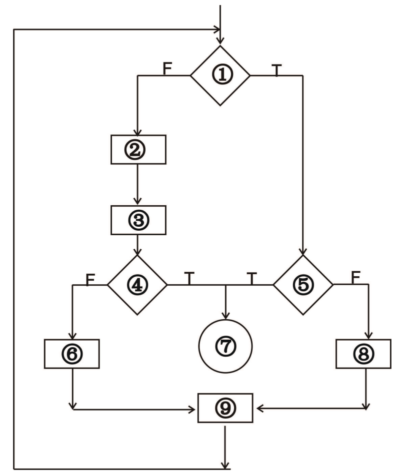

FIG. 1. (1) Verify the state is pure or not; (2) Decompose ρ into spectral decomposition; (3) For every factors a, b of the rank R such that $a\times b = R$, construct $(n+a+b)$-partite pure state $|\Phi\rangle^{ab}$ in Eq. (9); (4) For every permutation $\{q_1, q_2, \cdots, q_n\}$ and $0<\ell<n$, verify the rank of the coefficient matrix $r(C_{q_1\cdots q_\ell n+1\cdots n+a}(|\Phi\rangle^{ab})) = 1$ or not; (5) For every permutation $\{q_1, q_2, \cdots, q_n\}$ and $0<\ell<n$, verify the rank of the coefficient matrix $r(C_{q_1\cdots q_\ell}(|\Phi\rangle)) = 1$ or not; (6) Decompose ρ into the product of $(q_1\cdots q_\ell)\times(q_{\ell+1}\cdots q_n)$ bipartite state $\rho = \rho_1\otimes\rho_2$; (7) ρ or $|\Phi\rangle$ must be genuine correlated state; (8) Decompose $|\Phi\rangle$ into the product of $(q_1\cdots q_\ell)\times(q_{\ell+1}\cdots q_n)$ bipartite state $|\Phi\rangle = |\phi\rangle_{q_1\cdots q_\ell}\otimes|\phi'\rangle_{q_{\ell+1}\cdots q_n}$; (9) Input $\rho_1, \rho_2, |\phi\rangle_{q_1\cdots q_\ell}, |\phi'\rangle_{q_{\ell+1}\cdots q_n}$ into (1) respectively. "T" represent true, and "F" is false.

RELATIONSHIP BETWEEN THE HIGHER RANK (LARGER THAN ONE) OF THE COEFFICIENT MATRIX AND NO GENUINE CORRELATED STATE.

It is shown by Li *et al.* [18] that the rank of any coefficient matrix is invariant under stochastic local operations and classical communication (SLOCC). For the state with coefficient matrix of higher rank, we give the following theorem which indicates that the rank is a kind of measure of genuine multipartite correlations.

Theorem 4. *For a n particle pure state $|\psi\rangle_{1\cdots n}$, and a permutation $\{q_1, q_2, \cdots, q_n\}$, if the rank of the coefficient matrix $r(C_{q_1\cdots q_\ell}(|\psi\rangle_{1\cdots n})) = k, (k > 1)$, then $|\psi\rangle_{1\cdots n}$ can be expressed as a sum of k $(q_1\cdots q_\ell)\times(q_{\ell+1}\cdots q_n)$ bipartite product states, each of them has no genuine correlation.*

Proof. We first consider the pure state. Suppose the permutation $\{q_1, q_2, \cdots, q_n\}$ and $\ell(0<\ell<n)$ such that $r(C_{q_1\cdots q_\ell}(|\psi\rangle_{1\cdots n})) = k$, we rewrite $|\psi\rangle_{1\cdots n} = \sum_{i=0}^{2^\ell-1}\sum_{j=0}^{2^{n-\ell}-1} a_{ij}|i\rangle\otimes|j\rangle$, where $|i\rangle$ represents the $q_1\cdots q_\ell$ qubit and $|j\rangle$ represents the $q_{\ell+1}\cdots q_n$ qubit. The coefficient matrix $C_{q_1\cdots q_\ell}(|\psi\rangle_{1\cdots n})$ is given as $(a_{ij})_{2^\ell\times2^{n-\ell}}$. Denote each column vector of matrix as β_j

$$\beta_j = (a_{0j}, \cdots, a_{2^\ell-1j})^T, j = 0, \cdots, 2^{n-\ell}-1,$$

where T is the transpose of the vector, for convenience, we suppose $\beta_0, \cdots, \beta_{k-1}$ is linear independent, and the rest

of columns are the linear combination of the preceding k columns,

$$\beta_j = \sum_{v=0}^{k-1} t_{vj}\beta_v,\ j = k, \cdots, 2^{n-\ell} - 1,$$

which means

$$a_{ij} = \sum_{v=0}^{k-1} t_{vj} a_{iv}, \tag{11}$$

where $i = 0, \cdots, 2^\ell - 1$, $j = k, \cdots, 2^{n-\ell} - 1$.

We decompose $|\psi\rangle_{1\cdots n}$ into two terms, $|\psi\rangle_{1\cdots n} = |\psi\rangle_1 + |\psi\rangle_2$, where $|\psi\rangle_1 = \sum_{j=0}^{k-1}(\sum_{i=0}^{2^\ell-1} a_{ij}|i\rangle) \otimes |j\rangle$, and $|\psi\rangle_2 = \sum_{j=k}^{2^{n-\ell}-1}(\sum_{i=0}^{2^\ell-1} a_{ij}|i\rangle) \otimes |j\rangle$. Inserting (11) into $|\psi\rangle_2$, we have

$$|\psi\rangle_2 = \sum_{v=0}^{k-1}\sum_{i=0}^{2^\ell-1}\sum_{j=k}^{2^{n-\ell}-1} t_{vj} a_{iv}|i\rangle \otimes |j\rangle, \tag{12}$$

change the letter v as j, and j as v, we have that

$$\begin{aligned} |\psi\rangle_2 &= \sum_{j=0}^{k-1}\sum_{i=0}^{2^\ell-1}\sum_{v=k}^{2^{n-\ell}-1} t_{jv} a_{ij}|i\rangle \otimes |v\rangle \\ &= \sum_{j=0}^{k-1}(\sum_{i=0}^{2^\ell-1} a_{ij}|i\rangle) \otimes (\sum_{v=k}^{2^{n-\ell}-1} t_{jv}|v\rangle), \end{aligned} \tag{13}$$

therefore,

$$|\psi\rangle_{1\cdots n} = \sum_{j=0}^{k-1}(\sum_{i=0}^{2^\ell-1} a_{ij}|i\rangle) \otimes (|j\rangle + \sum_{v=k}^{2^{n-\ell}-1} t_{jv}|v\rangle). \tag{14}$$

Eq.(14) means that $|\psi\rangle$ is a sum of k bipartite product states, which has no genuine correlation. It is shown that the sum of k rank one matrix must has rank no more than k, we obtain that $|\psi\rangle_{1\cdots n}$ can not be expressed as a sum of less than k $(q_1 \cdots q_\ell) \times (q_{\ell+1} \cdots q_n)$ bipartite product states. □

HOW THE ALGORITHM WORKS

To show how our algorithm works, we consider further some examples. We next study an example of the entanglement swapping. Four Bell states are given as

$$|\beta_{xy}\rangle = \frac{|0, y\rangle + (-1)^x|1, \overline{y}\rangle}{\sqrt{2}}, \tag{15}$$

where $x, y = 0, 1$ and $\overline{y}$ is the negation of y. We now construct a four parties entanglement state $|\Phi\rangle_{AB|CD}$ as follows

$$|\Phi\rangle_{AB|CD} = \frac{1}{2}\Sigma_{x,y=0}^1 |\beta_{xy}\rangle_{AB} \otimes |\beta_{xy}\rangle_{CD}. \tag{16}$$

We can find that the coefficient matrix $C(|\Phi\rangle_{AC})$ and $C(|\Phi\rangle_{AD})$ are given by

$$C(|\Phi\rangle_{AC}) = C(|\Phi\rangle_{AD}) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \tag{17}$$

which is rank 1. Therefore, by Theorem 1, $|\Phi\rangle_{AB|CD}$ has no genuine multipartite correlation and can be expressed as a product of $AC|BD$ and $AD|BC$ bipartite cut. In fact,

$$|\Phi\rangle_{AB|CD} = |\beta_{00}\rangle_{AC} \otimes |\beta_{00}\rangle_{BD} = |\beta_{00}\rangle_{AD} \otimes |\beta_{00}\rangle_{BC},$$

is the condition for the so called entanglement swapping similar as teleportation, that is often invoked in quantum information processes so that a long-distance entangled state is generated from the short-distance entangled states.

We next use our method to prove that Smolin state [21] has genuine multipartite correlation. The Smolin state is given by

$$\rho = \frac{1}{4}\Sigma_{x,y=0}^1 |\beta_{xy}\rangle_{AB}\langle\beta_{xy}| \otimes |\beta_{xy}\rangle_{CD}\langle\beta_{xy}|, \tag{18}$$

Eq. (18) is actually the spectral decomposition of ρ, and the rank of ρ is 4, which can be a factorization of 2×2 and 1×4, thus, the purification of ρ to $(4 + 2 + 2)-$qubit pure state is given as

$$|\Phi\rangle^{22} = \frac{1}{2}\Sigma_{x,y=0}^1 |\beta_{xy}\rangle_{AB} \otimes |\beta_{xy}\rangle_{CD} \otimes |x, \overline{x}\rangle_{EF} \otimes |y, \overline{y}\rangle_{GH}. \tag{19}$$

Explicitly, we have,

$$\begin{aligned} &|\Phi\rangle^{22} = \\ &\frac{1}{4}((|00\rangle + |11\rangle)_{AB} \otimes (|00\rangle + |11\rangle)_{CD} \otimes |01\rangle_{EF} \otimes |01\rangle_{GH} \\ &+(|01\rangle + |10\rangle)_{AB} \otimes (|01\rangle + |10\rangle)_{CD} \otimes |01\rangle_{EF} \otimes |10\rangle_{GH} \\ &+(|00\rangle - |11\rangle)_{AB} \otimes (|00\rangle - |11\rangle)_{CD} \otimes |10\rangle_{EF} \otimes |01\rangle_{GH} \\ &+(|01\rangle - |10\rangle)_{AB} \otimes (|01\rangle - |10\rangle)_{CD} \otimes |10\rangle_{EF} \otimes |10\rangle_{GH}). \end{aligned} \tag{20}$$

It is an 8-qubit pure state. By Theorem 3, we first check whether the rank of the following 7 matrices

$$C(|\Phi\rangle^{22}_{AEF}), C(|\Phi\rangle^{22}_{BEF}), C(|\Phi\rangle^{22}_{CEF}), C(|\Phi\rangle^{22}_{DEF}),$$

$$C(|\Phi\rangle^{22}_{ABEF}), C(|\Phi\rangle^{22}_{ACEF}), C(|\Phi\rangle^{22}_{ADEF})$$

are equal to one or not (it is not difficult to find that $r(C(|\Phi\rangle^{22}_{AEF})) = r(C(|\Phi\rangle^{22}_{AGH})) = r(C(|\Phi\rangle^{22}_{BCDEF})))$. It will be tedious for direct verifying since they are 8×32 or 16×16 order matrices. In the following, we only need to show that there is a nonzero 2×2 minor for each matrix. We first give a detailed construction of the nonzero 2×2 minor of $C(|\Phi\rangle^{22}_{AEF})$. From (20), we select 2 terms and write them into $AEF|BCDGH$ bipartite cut,

$$|00\rangle_{AB}|00\rangle_{CD}|01\rangle_{EF}|01\rangle_{GH} = |001\rangle_{AEF}|00001\rangle_{BCDGH},$$

$$|11\rangle_{AB}|00\rangle_{CD}|01\rangle_{EF}|01\rangle_{GH} = |101\rangle_{AEF}|10001\rangle_{BCDGH}.$$

By (20) the coefficients of $|001\rangle_{AEF}|00001\rangle_{BCDGH}$ and $|101\rangle_{AEF}|10001\rangle_{BCDGH}$ are $\frac{1}{4}$, we further construct two terms, $|101\rangle_{AEF}|00001\rangle_{BCDGH}$ and $|001\rangle_{AEF}|10001\rangle_{BCDGH}$, which lie on the cross of the row and column of the forging two terms and do not belong to the terms of (20), the coefficients are 0.

Therefore, the four coefficients constitute a nonzero 2×2 minor of $C(|\Phi\rangle^{22}_{AEF})$, which means $r(C(|\Phi\rangle^{22}_{AEF}))\geq 2$. It is shown that for any permutation and combination of different parties, we can always find a nonzero minor of the coefficient, the submatrices we select are all $\frac{1}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

The purification of ρ to $(4+1+4)-$qubit pure state is given as

$$
\begin{aligned}
|\Phi\rangle^{14} = \frac{1}{4}((|00\rangle+|11\rangle)_{AB}&\otimes(|00\rangle+|11\rangle)_{CD}\otimes|1\rangle_E\otimes|1000\rangle_{FGHI}\\
+(|01\rangle+|10\rangle)_{AB}&\otimes(|01\rangle+|10\rangle)_{CD}\otimes|1\rangle_E\otimes|0100\rangle_{FGHI}\\
+(|00\rangle-|11\rangle)_{AB}&\otimes(|00\rangle-|11\rangle)_{CD}\otimes|1\rangle_E\otimes|0010\rangle_{FGHI}\\
+(|01\rangle-|10\rangle)_{AB}&\otimes(|01\rangle-|10\rangle)_{CD}\otimes|1\rangle_E\otimes|0001\rangle_{FGHI}),
\end{aligned}
\tag{21}
$$

for any permutation and combination of particles "$ABCD$". Since Smolin state is highly symmetric, we can always find a nonzero 2×2 minor of the coefficient matrix with combining "E", by Theorem 3, we have that the Smolin state has genuine multipartite correlation.

SUMMARY AND DISCUSSION.

In this paper we provide an efficient method for detecting genuine multipartite correlations of arbitrary n-qubit system in terms of the rank of coefficient matrices. The necessary and sufficient condition for a state with genuine correlation can be related to the rank of the coefficient matrix of a pure state, which is shown to be invariant under SLOCC. The proposed measure satisfies those general postulates raised by Bennett *et al.* in Ref.[12].

We would like to remark that the proposed method is essentially algorithmic. Thus one can follow the algorithm systematically to check for the genuine multipartite correlation. We have also provided several interesting examples for pure or mixed state cases to illustrate the power of the approach, namely symmetric pure states, a state related to entanglement swapping, a mixed state of W state and GHZ state (which has genuine multipartite correlation), and the Smolin state (which also has genuine multipartite correlation).

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* libobeijing2008@gmail.com
† hfan@iphy.ac.cn

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[22] Suppose $|\psi\rangle_{1\cdots n}=|\phi\rangle_{q_1\cdots q_\ell}\otimes|\phi'\rangle_{q_{\ell+1}\cdots q_n}=|\varphi\rangle_{p_1\cdots p_{\ell'}}\otimes|\varphi'\rangle_{p_{\ell'+1}\cdots p_n}$, then we use the form $|\varphi\rangle_{p_1\cdots p_{\ell'}}\otimes|\varphi'\rangle_{p_{\ell'+1}\cdots p_n}$ trace out the $q_1\cdots q_\ell$-particles is also a pure state, which means that $|\psi\rangle_{1\cdots n}$ can be further decompose to a product of the intersection of $q_1\cdots q_\ell$ and $p_1\cdots p_{\ell'}$-particles, here $q_1\cdots q_n$, $p_1\cdots p_n$ are permutations of $1,\cdots,n$.